Classification and Related Methods of Data Analysis
H.H. Bock (Editor)



# GOALS, ACHIEVEMENTS, AND LIMITATIONS OF MODERN FUNDAMENTAL MEASUREMENT THEORY

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The goal posed is: To understand fully those qualitative situations, corresponding to possible empirical observations, that can be represented faithfully (isomorphically) in some numerical system that includes ≥. The key idea turns out to be a classification of symmetries (automorphisms) of structures which, for highly symmetric (homogeneous) ones with representations on the continuum, is very simple. It leads to a uniform scheme for finding numerical representations. Generalizations are needed for structures of three types: non-homogeneous ones, those leading to geometric representations, and those leading to random variable representations.

## 1. GOALS OF THE MEASUREMENT THEORY ENTERPRISE

A limited (though not so limited) goal is: To understand fully those qualitative situations, corresponding to possible empirical observations, that can be represented faithfully (isomorphically) in some numerical system that includes numerical inequality, ≥.

Let me unpack these five phrases:

"To understand fully": means to classify the inherently different possibilities and to work out in considerable detail the properties of each.

"Those qualitative situations": means capturing the properties of the situation using mathematics that does not presume numbers - it uses the concepts of sets and relations.

"Corresponding to possible empirical observations": means that each primitive relation we postulate should potentially correspond to an empirical observation.

"That can be represented faithfully (isomorphically)": means to seek representations of each qualitative system that preserve the structure in that system; there is to be no gain or loss of information in representing the structure.

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"In some numerical system that includes numerical inequality, ≥,": means just what it says, and that implies that one of the primitives must be a qualitative ordering - reflecting an attribute that varies in degree or amount. This restriction to numerical representations is why I refer to this as a limited goal. More ambitious goals will be cited later.

A major motive behind the limited goal is the existence of powerful analytic techniques - including calculus, functional equations, and dimensional analysis - when variables and laws are stated in numerical form. I do not intend to suggest that no powerful mathematical techniques are available in the absence of numbers, for that surely is false.

A fruitful approach to these general questions is to explore the entire range of possible scale types that can arise, where by *scale type* we mean the kind of things S.S. Stevens [1, 2] was talking about when he classified measurement representations as (nominal), ordinal, interval, and ratio. The origins of Stevens' classification were the debates prompted by the physicist and philosopher of science N.R. Campbell and the 1930's Commission of the British Association for the Advancement of Science which was asked to look into the question of whether fundamental measurement is possible, in principle, for psychological attributes. The physicists on the Commission concluded: No, such fundamental measurement is not possible in psychology. Their syllogism was:

1. Fundamental measurement is possible if and only if there is an ordering and an operation of combining that are faithfully represented by ≥ and +. (Such structures are called extensive).

2. Psychology has none such.

3. Ergo, fundamental measurement is not possible in psychology (or any science lacking extensive operations).

The trouble with the syllogism centered on the general hypothesis of what constitutes fundamental measurement. No one involved with the Commission seemed especially sensitive to counter-examples to this view, although I am unclear how they thought representations involving averages got into the picture. S.S. Stevens argued that the important issue was not the additive representation, per se, but rather the comparative uniqueness that the underlying structure imposed on the possible additive representations. This uniqueness is described by the group of transformations that relate the various representations. Recognizing this and taking into account the groups that had arisen in practice led to his 1946 classification of scale types - ordinal, interval, and ratio.

These ideas were explored in a Harvard seminar during the early 1940s that involved prominent philosophers of science, mathematicians, and physicists. One gains the impression that they brought to Stevens' attention the importance in both physics and mathematics of the groups of transformations underlying these three scales.

What is needed to fill out Stevens' point of view? Most succinctly:

we need to lay out fully, in terms of a concept of scale type, the entire range of possibilities for numerical measurement. This really divides into five subprojects.

(i) We need to know what is meant in general by the concept of scale type, and to characterize all of the possible scale types. In particular, why are ratio, interval, and ordinal scales so important, and are there other types to be considered?

(ii) Given a particular scale type, we need to characterize the entire set of numerical structures exhibiting that scale type. These numerical structures are of interest since they are the only candidates for possible measurement representations. For example, was Campbell correct in his belief that $\langle Re^+, \geq, + \rangle$ (meaning the positive real numbers, $Re^+$, together with their natural order, $\geq$, and addition, +) is the sole candidate for ratio scaling of an empirical operation?

(iii) Given answers to the first two questions, we must seek the qualitative (potentially, empirical) regularities that, when satisfied by the phenomenon under study, lead to these representations. That is, we attempt to axiomatize the qualitative systems corresponding to the possible representations?

(iv) It is also important to explore the extent to which one-dimensional measurement structures can be coupled with conjoint (factorial) ones in such a way as to maintain the structure of units typical of classical physics. When

(a) the one-dimensional structure has an extensive operation, i.e., one that can be represented additively, and

(b) the conjoint structure can be represented multiplicatively,

then from physics we know that in many cases there is an empirical interlock between the two structures leading to a very simple pattern: the unit of the conjoint structure is the product of powers of the units of the components (for example: energy has the units of mass times velocity squared). The main problem is to characterize the other situations where this simple pattern is also adequate.

(v) And finally, given a structure of some scale type, we would like to know when a statement formulated either in terms of the primitives of the system or in terms of its numerical representation shall be judged to be "meaningful," and so is capable of being either true or false for that structure. In particular, what philosophically sound justifications can be given for the invariance conditions often invoked in meaningfulness arguments used in dimensional analysis, in geometry, and in discussions of the applicability of statistical methods to measurement?

Historically, these five problems were not worked on in the order listed. Early on, the focus was mostly on (iii) - axiomatizations of specific structures such as subjective expected utility, probability, and conjoint measurement - and on (iv) - the extent to which these new structures could, in principle be incorporated into the physical system of units. Important results about (i), (ii), and (v) have arisen only in the past decade. I will not attempt to deal with (v) - meaningfulness - here.

## 2. ACHIEVEMENTS IN THE HOMOGENOUS CASE

A further restriction on the limited goal stems from the fact that our understanding is fairly complete only when we restrict ourselves to cases where the objects or events to be measured cannot be distinguished one from another by their properties.

Let me cite some examples where particular entities have distinguished properties:

1. In velocity, let $u \oplus v$ denote the velocity obtained by "adding" u to v. According to special relativity, for velocities u,v less than light, c, $u \oplus v > u$ and $u \oplus v > v$, but $u \oplus c = c = c \oplus u$.

2. In probability only the universal event $\Omega$ has the property that for every event A, $\Omega \cup A = \Omega$, and only the null event $\emptyset$ has the property that for every event A, $\emptyset \cap A = \emptyset$.

3. In a system with an operation of combining elements, a zero element e is that unique element such that for every other element x, $x \oplus e = e \oplus x = x$.

In each case, one element has a property not exhibited by any other element. When that is not the case - when the elements cannot be distinguished by their properties, but only by their identity - we say the structure is *homogeneous*. The first achievement was to capture in mathematical terms exactly what we mean by the concept of homogeneity. This was done by L. Narens in 1981 [3,4]. It is as follows:

An (ordered) *measurement structure* is a set of elements with relations on it, one of which is an order (technically, a simple order). I will simply speak of "structure" for such a relational system. Define a *symmetry* (physicist's term) or *automorphism* (mathematician's term) to be any structure preserving (isomorphic) mapping of a structure onto itself. (Examples: Any rotation of a circle about its center. Any 90 degree rotation of a square about its center. Multiplication of lengths by a fixed positive constant.) Then the structure is called *homogeneous* if and only if for each pair of elements in the structure, some symmetry takes one element into the other element.

Example: $< Re+, \gtrsim, + >$ has as its symmetries multiplication by positive real numbers. It is trivially homogeneous: for suppose $x > 0$, $y > 0$, then y is transformed into x under the symmetry $z = x/y > 0$ since $zy = (x/y)y = x$.

It is useful to partition the symmetries into two types: Symmetries having no fixed point (no element a that is transformed into itself) are called *translations*. Symmetries with at least one fixed point are called *dilations*. The identity transformation (every point is a fixed point) is considered a translation as well as a dilation.

Example: for interval scales, the symmetries are the transformations $x \to rx + s$, $r > 0$; the translations are those with $r = 1$, i.e., $x \to x + s$. The dilations are those with either $r \neq 1$, $s \neq 0$ or $r = 1$, $s = 0$.

Let us turn now to the four questions:

(i) *What scale types are possible?*

Stevens had it nearly right: For structures that

(a) can be mapped *onto* the real numbers,

(b) are homogeneous, and

(c) are unique in the sense that specifying a fixed finite number of values of a representation completely specifies it for all elements,

there are just three possibilities: ratio, interval, and another between these two. Nothing else is possible. (This result was gradually achieved in a series of papers: L. Narens, [3,4] and T. Alper, [5,6].) The transformations available are[1]:

- ratio: $x \to x + s$ (s any real)
- discrete interval: $x \to k^n x + s$ ($k > 0$ and fixed, n any integer, s any real)
- interval: $x \to rx + s$ ($r > 0$, s any real).

(ii) *What numerical structures go with the scale types?*

(a) For numerical structures with a binary operation (addition, averaging a examples), the answer is fully known (Luce and Narens [7]). For example, the most general interval scale operation is:

$$x \oplus y = ax + (1-a)y + b|x-y| \quad (0 < a < 1, -\tfrac{1}{2} < b < \tfrac{1}{2}).$$

This leads to a version of utility theory somewhat like, but somewhat different from, Kahnemann and Tversky's [8] prospect theory. It handles many of the empirical anomalies, but it has not really been directly tested.

(b) Much more generally, the target representations are a class of numeric structures on $Re^+$ called *real unit structures* that have the property that the translations (symmetries with no fixed point) appear as multiplication by a positive constant (Luce, [9,10]).

(iii) *What qualitative structures map into numerical ones?*

The most general answer known for the homogeneous case is any structure whose translations act formally like multiplication by positive constants [10]. In practice, this means that for a specific structure, one should attempt to characterize and study its translations. If they can be shown to be homogeneous and to act like multiplication by a constant, then we know the structure has a representation. (Actually, constructing it is something else again.) For example, in structures with operations the study of the translations has been done fully. In particular, we know the conditions when any structure with a operation has a numerical representation, and we can characterize fully those structures that are homogeneous. For positive operations ($x \oplus y > x$ and $> y$), a constructive method is known for finding the representation (Luce, Krantz, Suppes, and Tversky, [11]).

(iv) *Fitting into the scheme of physical units*

The basic issue is as follows: Suppose we have a factorial structure - an ordering $\succsim$ of stimuli with two factors, drawn from two sets A and P, respectively. (Physical example: ordering of masses generated by containers (A) and substances (P).) And suppose we have a measurement structure on one of the factors, A, that has a numerical representation, say $\varphi$. (Physical example continued: ordinary measure of volume of containers.) Under what conditions will the conjoint structure have a numerical representation of the form $\varphi^{\rho}\psi$, where $\rho$ is a constant and $\psi$ is a numerical mapping of the second factor?

The general answer is really quite simple (although it took us quite a spell to get there). First, the structure on A must relate nicely to the conjoint ordering - technically, it is said to distribute in it. This means the following: Two n-tuples of elements from A, $(a_1, \ldots, a_n)$, and $(b_1, \ldots, b_n)$, are said to be *similar* (relative to the given conjoint structure) if there are elements p and q in P such that for each $i = 1, \ldots, n$, $(a_i, p) \sim (b_i, q)$. The structure on A is said to *distribute* in the conjoint structure if and only if each of its primitive defining relations S has the property: if $(a_1, \ldots, a_n)$ is in S and $(b_1, \ldots, b_n)$ is similar to $(a_1, \ldots, a_n)$, then $(b_1, \ldots, b_n)$ is also in S. The second property assumed is that the structure on A has a real unit representation, i.e., its translations act like multiplication by a constant. These two conditions along with solvability in the conjoint structure force the product of powers representation to exist [10].

## 3. LIMITATIONS ON WHAT HAS BEEN DONE

Within the limited goal of mapping into the ordered real numbers, the major limitation is: Almost all of our strong results are restricted to the homogeneous case. They do not apply to systems with either upper bounds (e.g., velocity, probability, and quite possibly sensory attributes) or zero elements. In particular, we do not really understand qualitatively how bounded scales, such as velocity, are tied into the system of units. Distribution as currently defined does not hold in that case.

In working with non-homogeneous structures, efforts probably should be restricted at first to structures, like the examples mentioned, in which homogeneity holds on both sides of a single distinguished element. I am not sure how difficult it will turn out to be to develop an understanding of these non-homogeneous cases. At present we have fragmentary results about some of the questions and none about others.

There are two major directions for expanding the investigations beyond the limited goal:

(i) *Geometric representations*

These clearly are of much interest for psychology, especially as they arise both in multidimensional scaling and in cognitive theory, and the social sciences more generally. Geometries and their axiomatization form a huge mathematical field. However, many of those results do not seem directly relevant to psychology because our empirical primitives do not correspond to points, lines, and incidence relations. Some axiomatic work was carried out 15 years ago, but the topic is far from fully mined.

(ii) *Random variable representations*

Error of measurement is handled only clumsily using numerical representations. And so it is really very difficult to be certain whether a body of empirical data reject a particular measurement model. A far more satisfactory state of affairs would be representations not into numbers but into families of random variables, which in the limiting case of no variability would reduce to known numerical models. The axiomatization would have to lead to a characterization of the distribution functions of the random variables, including of course the way random variables relate (e.g., $Z = X + Y$ in the simplest case).

Despite the fact we have been aware of this need for at least 25 years, not a single example has been forthcoming. There is to my knowledge no axiomatization of any family of random variables - the normal, the gamma, etc. These families arise from convolutions and asymptotic theorems, not from axiomatizations of properties of the families.

## 4. CONCLUSIONS

We have come a long way toward understanding homogeneous structures. We know a lot about

(a) the classification of structures by scale type, and

(b) the properties that an ordered qualitative structure must satisfy in order to have a numerical representation.

Moreover, results in this case are really quite simple and neat; however, the proofs so far developed tend not to be easy.

Much, however, remains to be done including at least the following:

1. A comparable development for non-homogeneous structures, especially those that are homogenous on either side of an isolated element.

2. A comparable development, again carried out first for the homogeneous case, leading to geometric representations; this should involve primitives suitable to behavioral and social science.

3. A comparable development, to be carried out first for the homogeneous case, leading to random variable representations.

FOOTNOTE:

[1]When the representation is on the entire real numbers, the transformations are as shown, the first of which is technically the translation group and one should properly refer to the representations as forming a difference scale. If an exponential transformation $u = e^x$ is made, so the representation is onto the positive real numbers, then the transformations from one representation to another become $u \to tu$, where $t = e^s > 0$. These are the similarity transformations and constitute the usual definition of a ratio scale. Analogously, the affine transformations of the interval scale become the power transformations $u \to tu^r$, and the representations are said to form a log-interval scale (because a log transformation takes them into the usual interval scale representations). Thus, holding the domain fixed, difference and interval scales form a natural pair as do ratio and log-interval scales. However, there is some tendency to speak of the one parameter case as ratio and the two parameter one as interval without regard to the domain, Re or $Re^+$. The greatest danger in doing this is to think that the ratio case is the special case of the affine transformation gotten by setting $s = 0$ rather than setting $r = 1$.

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